△ Invariant Measure on Lie Group

Flanders: differential forms

- $\Omega(\pi) = e^{i\pi\cdot T}$ $\pi^a \equiv$ Lie parameters

  $a = 1, \dots, N$

- $d\pi \to \Omega^{-1} d\Omega \equiv \Omega^{-1}\partial_a \Omega \cdot d\pi^a$ Cartan form

$$\equiv i T^a f^a = i T^a f^{ab}(\pi) \cdot d\pi^b$$

$$\boxed{d\mu_\pi \equiv f^{a_1} \wedge \dots \wedge f^{a_N} \, \frac{\varepsilon^{a_1 \dots a_N}}{N!}}$$

Haar Measure

$$f^a = f^{ab} \, d\pi^b$$

$dx \wedge dy$ 

$dx \wedge dy \wedge dz$

$$= \det(f) \cdot d\pi^1 \wedge \dots \wedge d\pi^N$$

- $d\mu_\pi$ invariant under

  - left G action: $\Omega(\pi) \to g\,\Omega(\pi) \equiv \Omega(\pi_L(g))$
  - right G action: $\Omega(\pi) \to \Omega(\pi)\,g = \Omega(\pi_R(g))$

Faddeev-Popov : toy example

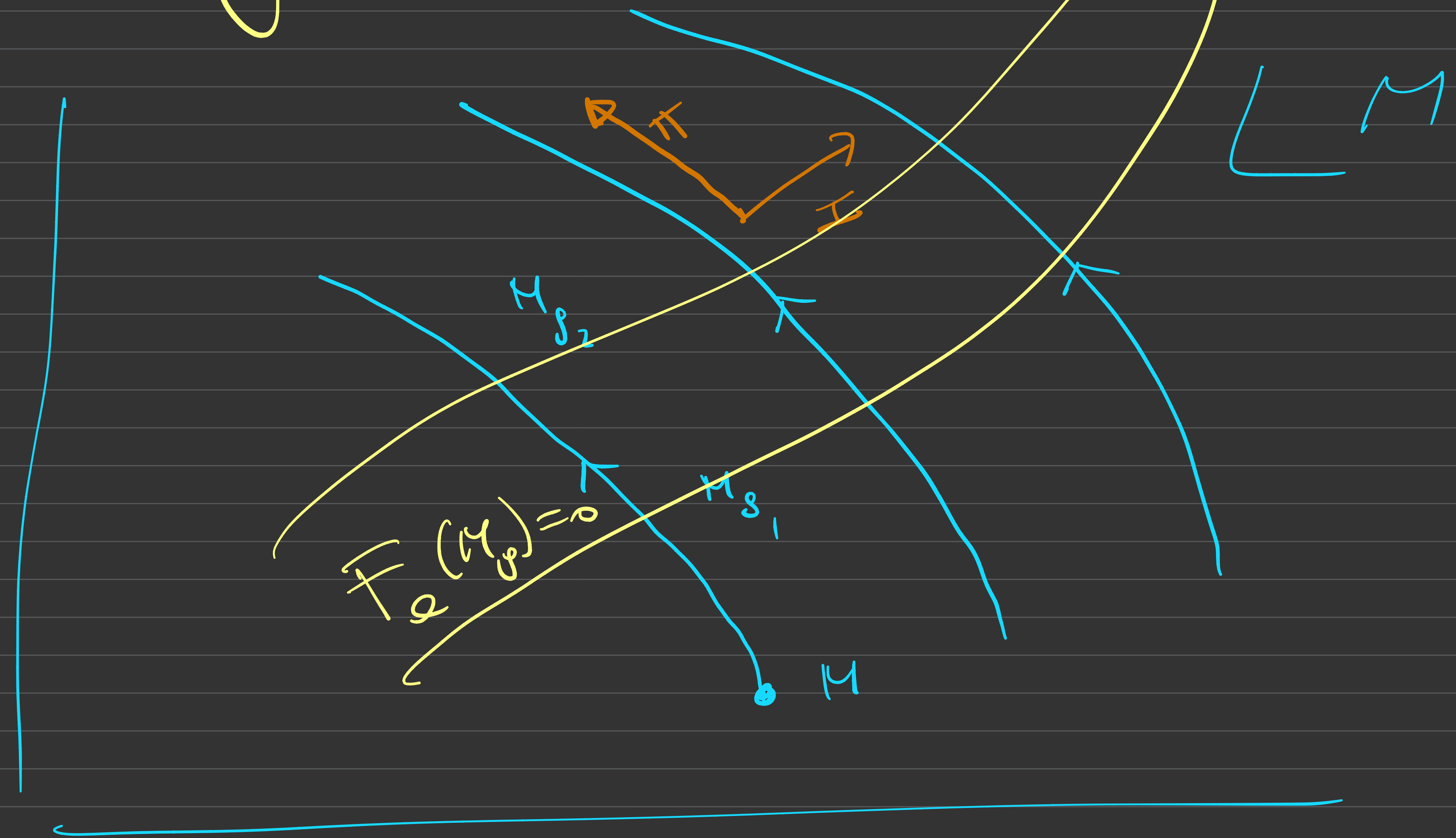
- G = unitary (compact) Lie Group

- M_α = representation : $M_\alpha \longrightarrow (M_g)_\alpha = D(g)_{\alpha\beta} M_\beta$

$$M_g = D(g) M$$

- $g(\pi) \equiv e^{i\pi \cdot T}$ $\qquad d\mu_\pi \equiv$ Haar measure

- $Z = \int dM \, e^{-S(M)}$



- $S(M_g) = S(M)$
- $dM_g = dM$

- want to write: $dM \sim \underbrace{d\mu_\pi}_{G\text{-orbits}} \times \underbrace{dI}_{G\text{-invariants}}$

• Consider $\qquad F_a(M) \qquad a = 1, \dots, N$

hp

$$\begin{cases} \bullet \ \forall M \ \exists \bar{g} \mid F_a(M_{\bar{g}}) = 0 \qquad \forall A \\ \bullet \ \dfrac{\partial}{\partial \pi_b} F_a(M_{g(\pi)}) \Big|_{\pi=0} \equiv \Delta_{ab}(M) = \text{invertible} \end{cases}$$

th

$$\underbrace{\int d\mu_\pi \prod_a \delta\left(F_a(M_{g(\pi)})\right) \det \Delta(M_{g(\pi)})}_{M\text{- and } F\text{- independent} \ \equiv \ 1} = \blacksquare$$

• <u>proof</u> $\quad g(\pi) \equiv g(\tilde{\pi})\,\bar{g} \qquad M_{\bar{g}} \equiv \bar{M}$

$$\left(M_{\bar{g}}\right)_{g(\tilde{\pi})} = D(g(\tilde{\pi}))\, M_{\bar{g}} = D(g(\tilde{\pi}))D(\bar{g})M = D(g(\pi))\, M = M_{g(\pi)} = \left(M_{\bar{g}}\right)_{g(\tilde{\pi})} = \bar{M}_{g(\tilde{\pi})}$$

$$\blacksquare = \int d\mu_{\tilde{\pi}} \prod_a \delta\left(F_a\left(\bar{M}_{g(\tilde{\pi})}\right)\right) \det \Delta\left(\bar{M}_{g(\tilde{\pi})}\right)$$

$$\bar{M}_{g(\tilde{\pi})} = e^{i\tilde{\pi}\cdot T}\,\bar{M} = \bar{M} + i\,\tilde{\pi}_b\,(T_b M) + O(\tilde{\pi}^2)$$

$$F_a\left(\bar{M}_{g(\tilde{\pi})}\right) = F_a(\bar{M}) + \partial_\beta F_a(\bar{M}) \cdot i\,\tilde{\pi}_b \left(T_b \bar{M}\right)_\beta + \dots$$

$$= \underbrace{i\,\partial_\beta F_a\,(T_b \bar{M})_\beta}_{} \cdot \tilde{\pi}_b + O(\tilde{\pi}^2)$$

$$\equiv \quad \Delta_{ab}(\bar{M}) \cdot \tilde{\pi}_b \quad + O(\tilde{\pi}^2)$$

$$= \int d\mu_{\tilde{\pi}} \prod_a \delta\left(\Delta_{ab}(\bar{M})\, \tilde{\pi}_b + O(\tilde{\pi}^2) \right) \det \Delta\left(\bar{M}_{g(\tilde{\pi})}\right)$$

$$= \int d^N \tilde{\pi} \prod_a \delta\left(\Delta_{ab}(\bar{M})\, \tilde{\pi}_b + O(\tilde{\pi}^2) \right) \det \Delta(\bar{M}) \left[1 + O(\tilde{\pi}) \right]$$

$$= \frac{1}{\det \Delta(\bar{M})} \cdot \det \Delta(\bar{M}) = 1$$

$$\int dM\, e^{-S(M)} = \int dM\, e^{-S(M)} \int d\mu_\pi\, \delta(F(M_{g(\pi)}))\, |\Delta(M_{g(\pi)})|$$

$$= \int dM\, d\mu_\pi\, e^{-S(M)}\, \delta(F(M_{g(\pi)}))\, |\Delta(M_{g(\pi)})|$$

$$= \int dM_{g(\pi)}\, d\mu_\pi\, e^{-S(M_g)}\, \delta(F(M_g))\, |\Delta(M_g)|$$

$$= \int d\mu_\pi\, dM'\, e^{-S(M')}\, \delta(F(M'))\, |\Delta(M')|$$

$$= \underbrace{\int d\mu_\pi}_{V_G} \times \underbrace{\int dM\, e^{-S(M)}\, \delta(F(M))\, |\Delta(M)|}_{\text{integral over invariants}}$$

$$= V_G \times \underbrace{Z_I}_{} = Z_I = \int dM\, e^{-S(M)}$$

independent on choice of F

- $F_a(M) \longrightarrow F_a(M) - B_a \Longrightarrow$ same result

$$Z_I = \int \mathcal{N}\, dB\, e^{-\frac{B^2}{2\xi}} \int dM\, e^{-S(M)} \delta(F(M)-B)\, |\Delta(M)|$$

$\mathcal{N} = \left(\frac{1}{2\pi\xi}\right)^{\frac{N}{2}}$

$$= \mathcal{N} \int dM\, e^{-S(M) - \frac{F(M)^2}{2\xi}} |\Delta(M)|$$

$$|\Delta| = \left| \frac{\delta F_a}{\delta \pi_b} \right|$$

Quantum Gauge Theory

Symmetry $\hat{G} = \bigotimes_x G_x$

- naive path integral:

$$\underbrace{\int \mathcal{D}\psi \mathcal{D}A}_{\Longrightarrow \text{ gauge invariant}} \; e^{i \overbrace{\int \left\{ -\frac{1}{4g^2} G_{\mu\nu} G^{\mu\nu} + \mathcal{L}_M \right\}}} \equiv \int \mathcal{D}\psi \mathcal{D}A \, e^{iS[A,\psi]}$$

$\Longrightarrow$ gauge invariant

Ex $\quad DA \equiv \prod_{x,a,\mu} dA_\mu^a(x)$

$$A_\mu^a \longrightarrow A_\mu^a + \partial_\mu \alpha^a + f^{abc} A_\mu^b \alpha^c = A'^a_\mu$$

$$\Longrightarrow \quad DA \longrightarrow DA' \circ \prod_{x,a,\mu} \left(1 + f^{aac}\, \delta(0)\, \alpha^c(x)\right)$$

$$\frac{\delta A'^a_\mu(x)}{\delta A^d_\nu(y)} = \frac{\delta\left(A_\mu^a(x) + \partial_\mu \alpha^a(x) + f^{abc} A_\mu^b(x)\, \alpha^c\right)}{\delta A^d_\nu(y)}$$

$$= \delta^{ad} \delta^{\mu}_{\nu} \delta(x-y) + f^{adc} \delta^{\mu}_{\nu} \delta(x-y) \alpha^{c}(x)$$

$$= \mathbb{1}_{\alpha\beta} + \Delta_{\alpha\beta} + O(\alpha^2)$$

$$\downarrow$$

$$a, \mu, x$$

$$\det \frac{\delta A^{a}_{\mu}(x)}{\delta A'^{d}_{\nu}(y)} = \det(\mathbb{1} + \Delta) = 1 + \operatorname{tr}\Delta + \cdots$$

$$1 + \sum_{\mu} \sum_{a} \int dx \, f^{aac} \delta^{\mu}_{\mu} \delta(0) \alpha^{c}(x)$$

$$J = 1 + \int dx \, \underline{f^{aac}} \cdot 4 \, \delta(0) \, \alpha^c(x) = \underline{1}$$

$$DA\,D\psi\; e^{iS(A,\psi)} = D\mu_{\hat{G}}\; d\,\mathrm{Inv}\; e^{iS(\mathrm{Inv})}$$

$$\downarrow$$

$$V_{\hat{G}} = \prod_x V_G = V_G^{\infty} \equiv \underline{\underline{\text{ill defined}}}$$

<u>on the lattice</u>

$$V_{\hat{G}} = V_G^{N_{\text{links}}}$$

- $F^a(A(x))$ $\qquad a = 1, \dots, N$

- $\forall A \ \exists g \ | \ F^a(A_g) = 0 \quad \forall x, a$

- $\Delta_{ab}(A) \equiv \left. \dfrac{\delta F^a(A_{g(\alpha)})}{\delta \alpha^b} \right|_{\alpha^b = 0}$ = invertible

- $1 = \int d\mu_\alpha \prod_{x,a} \delta(F^a(A_{g(\alpha)})) \, \mathrm{Det}(\Delta(A_{g(\alpha)}))$

Ex 1: Lorentz gauge: $F^a = \partial^\mu A^a_\mu$

$$\frac{\delta}{\delta\alpha_b(y)} \partial^\mu \left(A_{g(\alpha)}(x)\right)^a_\mu = \frac{\delta}{\delta\alpha_b(y)}\left[\partial^\mu A^a_\mu + \partial^2\alpha^a(x) + f^{acd}\partial^\mu\left(A^c_\mu(x)\,\alpha^d(x)\right)\right]$$

$$= \delta^{ab}\,\partial^2_x\,\delta(x-y) + f^{acb}\,\partial^\mu\left(A^c_\mu\,\delta(x-y)\right)$$

$$= \partial^\mu \left(D_\mu\right)^{ab}\delta(x-y) \equiv \Delta_{ax,by}$$

$$\downarrow$$

$$\delta^{ab}\partial_\mu + f^{acb}A^c_\mu$$

Ex 2 Axial gauge : $F^a = A_3^a$

$$\frac{\delta\, [A_g(x)]_3^a}{\delta \alpha^b(y)} = \frac{\delta}{\delta \alpha^b(y)} \left[A_3^a(x) + \partial_3 \alpha^a(x) + f^{acd} A_3^c \alpha^d(x) \right]$$

$$= \delta^{ab} \partial_3 \delta(x-y) + f^{acb} A_3^c \delta(x-y)$$

$$\overset{A_3^a = 0}{=\!=} \delta^{ab} \partial_3 \delta(x-y)$$

- $\int D\psi\, DA\; e^{iS[A,\psi]}$ $\quad$ - $\int d\mu_\alpha\, \delta(F(A_\alpha))\, \det\Delta(A_\alpha)$

$$= \int D\psi_\alpha\, DA_\alpha\, d\mu_\alpha\; e^{iS[A_\alpha,\psi_\alpha]}\, \delta(F(A_\alpha))\, \det\Delta(A_\alpha)$$

$$= \underbrace{\int d\mu_\alpha}_{\infty} \times \underbrace{\int D\psi\, DA\; e^{iS[A,\psi]}\, \delta(F(A))\, \det\Delta(A)}_{\text{Integral over invariants}}$$

"independent" of choice of F

$$\int D\psi\, DA\; O_1(\psi, A) \cdots O_n(\psi, A)\, e^{iS}$$

$$\langle O_1 \cdots O_n \rangle$$

$$\int D\psi\, DA\; O_1(A,\psi) \cdots O_n\, e^{iS(A,\psi)} \int d\mu_\alpha\, \delta(F(A_\alpha))\, \Delta(A_\alpha)$$

- Similarly

$$\langle O_1(A,\psi)\cdots O_n(A,\psi)\rangle = \int DA\,D\psi\; O_1(A,\psi)\cdots O_n(A,\psi)\, e^{iS[A,\psi)}\;\delta(F(A))\,\det\Delta(A)$$

is independent of the choice for F if

$$O_i(A\,\psi) = O_i(A_\alpha,\psi_\alpha)$$

☆ Observables must be gauge-invariant

$B \equiv B_a(x)$

- $$\underbrace{\int dB_a \, e^{-\frac{i}{2\xi}\int B_a^2}}_{\text{const}} \cdot \underbrace{\int D\phi DA \, e^{iS'} \delta(F_a(A)-B_a) \det \Delta}_{B\text{-indep}}$$

$$= \int D\phi \, DA \left(\det \frac{\delta F_a}{\delta \alpha_b} \right) e^{iS - \frac{i}{2\xi} F_a^2}$$

$$\frac{\delta F_a(A(x))}{\delta \alpha_b(y)}$$

- $$\det \frac{\delta F_a(x)}{\delta \alpha_b(y)} = \int D\bar{\omega} D\omega \, e^{i\int \bar{\omega}_a \frac{\delta F_a}{\delta \alpha_b} \omega_b}$$

$\Delta(a,x;b,y)$

Faddeev-Popov ghost fields

$$\mathrm{Det}\,\Delta = \int dc_a\, d\bar{c}_b\; e^{\bar{c}_a \Delta_{ab} c_b}$$

$$\frac{\delta F_a(A(x))}{\delta \alpha_b(y)}$$

$$\det \frac{\delta F_a(x)}{\delta \alpha_b(y)} \equiv \int D\bar{\omega}\, D\omega\; e^{i\int \bar{\omega}_a \frac{\delta F_a}{\delta \alpha_b} \omega_b}$$

$$\Delta(a,x;b,y)$$

$$e^{i \sum_{a,b} \int dx\, dy\; \bar{\omega}_a(x)\, \Delta(a,x;b,y)\, \omega_b(y)}$$

- In the end the path integral is

$$\mathcal{Z}_\Psi = \int D\Psi\, DA\, D\bar{\omega}\, D\omega\; e^{iS' - \frac{i}{2\xi} F_a^2 + i\bar{\omega} \frac{\delta F}{\delta \alpha} \omega}$$

▲ **Ex** Lorenz gauge $\quad F^a = \partial^\mu A^a_\mu$

$$\frac{\delta F^a}{\delta \alpha^b} = \partial^\mu_x \underbrace{D^{ab}_\mu}_{\longrightarrow\ \delta^{ab}\partial_\mu + f^{acb} A^c_\mu}(x)\, \delta(x-y)$$

$$I_{gh} = \int \bar{\omega}_a(x)\, \partial^\mu_x D_\mu(x)^{ab}\, \delta(x-y)\, \omega_b(y)\, dx\, dy$$

$$= \int \bar{\omega}_a\, \partial^\mu D^{ab}_\mu\, \omega_b \quad dx$$

- A^a_μ Kinetic term

$$-\frac{1}{4g^2} G^a_{\mu\nu} G^{a\,\mu\nu} - \frac{1}{2g^2\xi}\left(\partial^\mu A^a_\mu\right)^2$$

$$\longrightarrow \frac{1}{2g^2}\left[-\partial_\mu A^a_\nu \partial^\mu A^{a\,\nu} + \left(1-\frac{1}{\xi}\right)\partial^\mu A^a_\mu \partial^\nu A^a_\nu\right] + \dots$$

$$\xrightarrow{\text{Fourier}} \frac{1}{2g^2} A^a_\nu(-p)\left[-p^2\eta^{\mu\nu} + \left(1-\frac{1}{\xi}\right)p^\mu p^\nu\right] A^a_\mu(p)$$

$$= \frac{1}{2g^2} A^a_\nu(-p)\, \Delta^{\mu\nu} A^a_\mu(p)$$

$$e^{iS} \longrightarrow e^{\frac{i}{2g^2} A \Delta A} \Longrightarrow \langle A A \rangle = i g^2 \Delta^{-1}$$

$$\Rightarrow \langle A^a_\mu A^b_\nu \rangle = i g^2 \delta^{ab} (\Delta^{-1})_{\mu\nu}$$

$$= \frac{i g^2 \delta^{ab}}{p^2} \left[-\eta_{\mu\nu} + (1-\xi) \frac{p_\mu p_\nu}{p^2} \right]$$

- Ghost propagator

$$\begin{cases} \int D\omega D\bar{\omega} \; e^{\bar{\omega}\Delta\omega} = \Delta \\ \int D\omega D\bar{\omega} \; \omega\bar{\omega} \; e^{\bar{\omega}\Delta\omega} = -1 \end{cases}$$

$$\longrightarrow \int \frac{\delta}{\delta\omega}\frac{\delta}{\delta\bar{\omega}}(1+\bar{\omega}\Delta\omega) = \Delta$$

$$\Longrightarrow \langle \omega\bar{\omega} \rangle = -\frac{1}{\Delta}$$

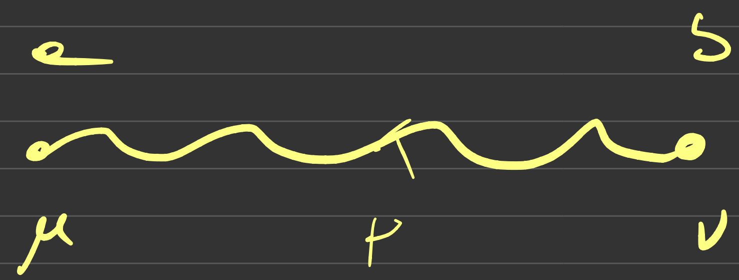
- in general

$$\int D\omega D\bar{\omega} \; e^{\bar{\omega}^a \Delta_{ab} \omega^b} \longrightarrow \langle \omega_a \bar{\omega}_b \rangle = -(\Delta^{-1})_{ab}$$

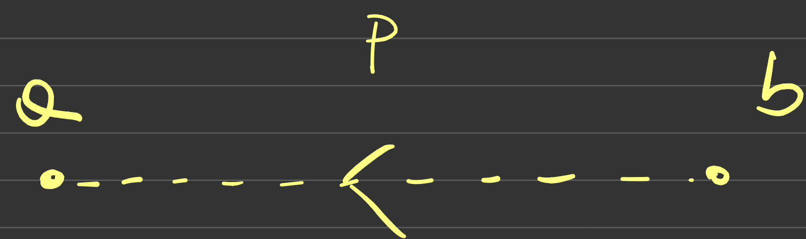
in our case $\bar{\omega}^a \Delta_{ab} \omega^b = \int i \, \bar{\omega}^a \partial^2 \delta_{ab} \omega^b$

$$\langle \omega_a \bar{\omega}_b \rangle = -\frac{\delta_{ab}}{i\partial^2} = \frac{-i\delta_{ab}}{p^2}$$

$\Rightarrow$



$$\frac{i\delta_{ab}}{p^2}\left(-\eta_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)$$

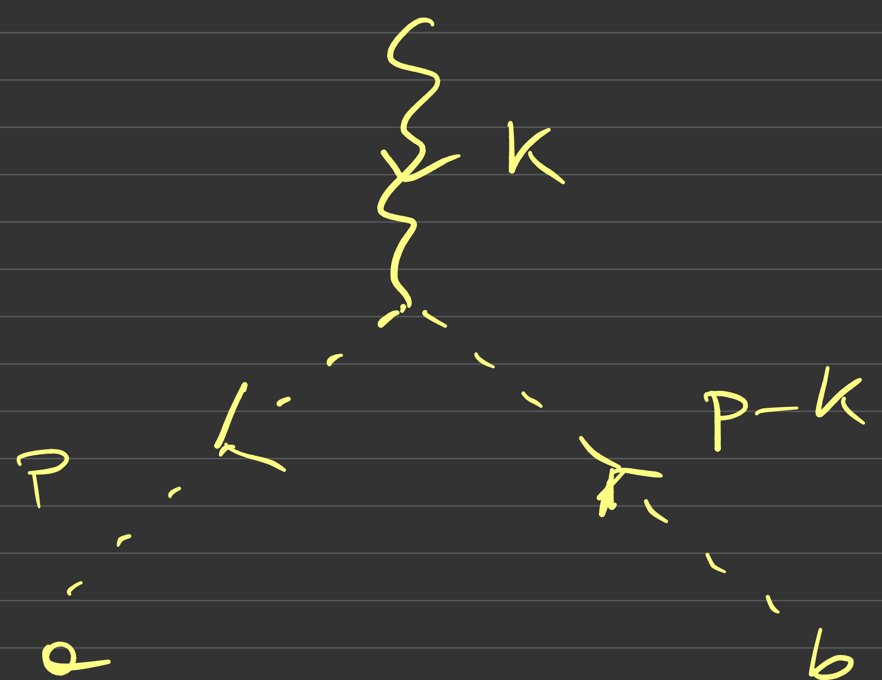


$$-\frac{i\delta_{ab}}{p^2}$$

- <u>Vertex</u>

$$i g \bar{\omega}^a f^{acb} \partial^\mu (A^c_\mu \omega^b)$$

$$= -i g \, \partial^\mu \bar{\omega}^a f^{acb} A^c_\mu \omega^b$$



$$g f^{acb} \quad p^\mu$$